Indian Statistical Institute, Bangalore

B. Math. Third Year

Second Semester - Differential Equations

Mid-Semester Exam   Duration: 3 hours   Date : February 24, 2016

Answer any five and each question carries 8 marks. Total marks: 40

1. (a) Prove that $Mdx + Ndy = 0$ is exact if and only if $M_y = N_x$.

   (b) Solve $(xy - 1)dx + (x^2 - xy)dy = 0$ (*Marks: 4*).

2. Find a necessary and sufficient condition under which the equation $M(x, y)dx + N(x, y)dy = 0$ has an integrating factor that is a function of $2x - y$ and in that case, find the integrating factor.

3. (a) Let y_1 and y_2 be two linearly independent solutions of $y''+P(x)y'+Q(x)y = 0$ on $[a, b]$. Show that $P = \frac{y_2 y_1'' - y_1 y_2''}{W(y_1, y_2)}$ and $Q = \frac{y_1' y_2'' - y_1'' y_2'}{W(y_1, y_2)}$ (*Marks: 5*).

   (b) Let y and y' be linearly independent solutions of $y'' + P(x)y' + Q(x)y = 0$ on $[a, b]$. Suppose y'' is also a solution. Prove that P and Q are constants.

4. (a) Reduce $x^2y'' + xpy' + q = 0$ to a linear equation with constant coefficients and use it to solve $x^2y'' + 2xy' - 12y = 0$ (*Marks: 3*).

   (b) Solve $y'' - 3y' + 2y = 14 \sin 2x - 18 \cos 2x$.

5. (a) Solve the system $x' = 3x - 4y$ and $y' = x - y$ (Marks: 4).

   (b) Solve $y' = x + y$, $y(0) = 1$ and compare it with the solution obtained by Picard's Theorem.

6. (a) Solve $(1+x)y' = py$, $y(0) = 1$ and prove $(1+x)^p = 1+\sum_{n\geq 1} \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n$ for $|x| < 1$ (*Marks: 4*).

   (b) Solve $y' = (1 - x^2)^{-1/2}$ and prove $\frac{\pi}{6} = 1/2 + 1/2 \frac{1}{3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{1}{5 \cdot 2^5} + \cdots$.

7. (a) Solve $y'' - 2xy' + 2py = 0$ and show that any solution is analytic on $\mathbb{R}$.

   (b) Find the solution and its radius of convergence of the ODE $y''+y'-xy = 0$, $y(0) = 1$, $y'(0) = 0$ (*Marks: 4*).